Using the Lagrangian Formulation to Derive and Numerically Predict the Motion of a Chaotic Double Pendulum

I. Abstract

The double pendulum is a canonical example of a deterministic yet chaotic mechanical system, exhibiting extreme sensitivity to initial conditions and complex, nonlinear motion. Traditional Newtonian approaches, while valid, become algebraically inefficient when applied to constrained multi-body systems such as the double pendulum. This paper examines how the Lagrangian formulation of classical mechanics provides a systematic framework for deriving the equations of motion of a double pendulum using generalised coordinates and energy-based methods. Starting from the principle of stationary action, the Lagrangian is constructed and used to obtain coupled, nonlinear differential equations governing the system's dynamics. Because these equations admit no closed-form solution, numerical integration methods are required to predict the pendulum's motion. Through this framework, the paper highlights the distinction between determinism and predictability, showing how chaotic behaviour fundamentally limits long-term prediction despite exact governing equations. Finally, the relevance of Lagrangian mechanics to broader applications, including aerospace engineering and multi-body dynamical systems, is briefly discussed.

II. Introduction

Many physical systems of practical and theoretical interest involve multiple interacting components subject to constraints. While Newtonian mechanics provides a complete description of classical motion, its force-based formulation can become unwieldy when applied to such systems, particularly when constraints introduce additional unknown forces. The double pendulum exemplifies this difficulty. Despite its simple physical construction, the double pendulum exhibits highly complex motion arising from nonlinear coupling between its components, making it a widely studied example of deterministic chaos.

The double pendulum consists of two masses connected by rigid rods, free to swing under gravity. At low energies, its motion appears regular and predictable; however, as energy increases, the system becomes extremely sensitive to initial conditions. Infinitesimal differences in starting angles or velocities can lead to dramatically different trajectories over time. This sensitivity poses a fundamental challenge to prediction, even though the system is governed by well-defined classical laws.

The Lagrangian formulation of mechanics offers a powerful alternative to the Newtonian approach for analysing such systems. Rather than focusing on forces, the Lagrangian method describes motion in terms of energy and generalised coordinates, allowing constraints to be incorporated directly into the mathematical structure of the problem. This makes it particularly well-suited to multi-degree-of-freedom systems such as the double pendulum, where the explicit treatment of constraint forces can be avoided altogether.

This paper investigates how the Lagrangian formulation can be used to derive the equations of motion of a double pendulum and how these equations enable numerical prediction of its motion. In doing so, it also examines the fundamental limits of predictability imposed by chaotic dynamics. By contrasting the Lagrangian approach with Newtonian mechanics and exploring its connection to numerical methods and Hamiltonian mechanics, the paper aims to demonstrate both the practical advantages and conceptual significance of the Lagrangian framework. The relevance of these ideas to applied fields such as aerospace engineering is also considered.

III. What is the Lagrangian

a) How the Lagrangian Formulation Emerged

Classical mechanics was originally formulated by Newton in terms of forces and accelerations. While Newton's laws provide a complete and correct description of motion, their direct application becomes increasingly inefficient for systems involving multiple bodies and constraints. In such cases, additional forces—such as tensions or reaction forces—must be introduced and carefully resolved, often leading to complex and cumbersome calculations.

An alternative perspective emerged through the development of variational principles in mechanics. Rather than describing motion as the result of forces acting at each instant, this approach characterises motion as a global property of a system's trajectory through time. Early work by Maupertuis and Hamilton introduced the concept of action, a scalar quantity defined as an integral over time, whose stationary value determines the actual path taken by a system. Joseph-Louis Lagrange formalised this idea into a general framework that eliminated the explicit use of forces altogether.

The key conceptual shift introduced by Lagrangian mechanics is the replacement of force-based reasoning with an energy-based description expressed in terms of generalised coordinates. Constraints that complicate Newtonian analyses can be incorporated naturally through an appropriate choice of coordinates, without introducing additional unknown forces. This reformulation is particularly advantageous for systems with multiple degrees of freedom, laying the foundation for modern analytical mechanics.

b) What Is the Lagrangian? (Mathematical Formulation)

In Lagrangian mechanics, the dynamics of a system are described by a single scalar function known as the Lagrangian, denoted by $\mathcal{L}$. For a wide class of classical systems, the Lagrangian is defined as the difference between the kinetic energy T and the potential energy V:

$$\mathcal{L}(q_i, \dot{q}_i, t) = T - V$$

Here, q_i represent generalised coordinates and $\dot{q}_i$ their corresponding time derivatives. The Lagrangian is not, by itself, the fundamental physical principle. Instead, it is used to construct the action S, defined as the time integral of the Lagrangian:

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

The central postulate of Lagrangian mechanics is the principle of stationary action, which states that the actual path taken by a system between two fixed points in time makes the action stationary with respect to small variations of the path. Applying this variational principle leads to the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

These equations replace Newton's second law as the equations of motion, providing a general and systematic method for analysing mechanical systems. The Lagrangian formulation is particularly powerful for constrained and multi-body systems, as it allows the equations of motion to be derived directly from energy expressions without explicit reference to forces.

IV. Newtonian vs Lagrangian Mechanics

Newtonian mechanics describes motion through forces acting on individual bodies, expressed by Newton's second law $F = ma$. In principle, this formulation can be applied to any classical mechanical system. However, when a system involves multiple bodies connected by constraints, the direct application of Newton's laws often becomes algebraically complex and conceptually inefficient. The double pendulum provides a clear example of these limitations.

In a Newtonian analysis of a double pendulum, separate free-body diagrams must be constructed for each mass. In addition to gravitational forces, unknown constraint forces such as tensions in the rods must be introduced. These forces are not of direct physical interest, yet they must be solved simultaneously in order to determine the motion. Furthermore, the resulting equations are strongly coupled and involve trigonometric terms arising from the changing geometry of the system. Although the Newtonian approach is mathematically valid, the complexity of the force-based formulation obscures the underlying structure of the dynamics and makes analytical progress difficult.

The Lagrangian formulation approaches the same problem from a fundamentally different perspective. Rather than focusing on forces, it describes the system in terms of generalised coordinates that directly encode the constraints of the motion. For the double pendulum, angular coordinates naturally capture the system's degrees of freedom, eliminating the need to consider constraint forces explicitly. The equations of motion are derived from scalar energy expressions, resulting in a systematic and compact procedure that scales efficiently with the number of degrees of freedom.

A key advantage of the Lagrangian approach is that constraints are incorporated implicitly through the choice of coordinates. This allows the equations of motion to be obtained without introducing auxiliary forces whose sole purpose is to enforce geometric relationships. As a result, the Lagrangian formulation provides not only a more efficient computational framework, but also clearer physical insight into the coupling and nonlinearity that give rise to chaotic behaviour in the double pendulum.

In this sense, Newtonian and Lagrangian mechanics are not competing theories, but alternative formulations of the same physical laws. Newtonian mechanics remains indispensable for many applications, particularly those involving unconstrained motion and direct force measurements. However, for constrained multi-body systems such as the double pendulum, the Lagrangian formulation offers a more natural and transparent framework. This structural advantage motivates its use in the subsequent derivation of the equations of motion and in the numerical analysis of chaotic dynamics.

V. Applying the Lagrangian to the Double Pendulum

a) System definition and coordinates

The double pendulum consists of two point masses, m_1 and m_2, connected by grid, massless rods of length l_1 and l_2. The first mass is attached to a fixed pivot, while the second mass is suspended from the first. The system is constrained to move in a vertical plane under the influence of gravity.

To describe the motion, generalised coordinates are chosen in the form of angular displacements θ_1 and θ_2, measured from the vertical. This choice naturally incorporates the geometric constraints imposed by the rigid rods and eliminates the need to consider tension forces explicitly. The configuration of the system at any time is therefore fully determined by these two angles, corresponding to two degrees of freedom.

Using this coordinate system, the Cartesian positions of the masses can be written as

$$\begin{aligned} x_1 &= l_1 \sin\theta_1, \\ y_1 &= -l_1 \cos\theta_1, \\ x_2 &= x_1 + l_2 \sin\theta_2, \\ y_2 &= y_1 - l_2 \cos\theta_2. \end{aligned}$$

These expressions provide the basis for computing the kinetic and potential energies required to construct the Lagrangian.

b) *Kinetic energy of the system*

The kinetic energy of the system is given by the sum of the kinetic energies of the two masses,

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2,$$

where v_1 and v_2 are the speeds of the first and second masses, respectively.

The velocity of the first mass is obtained by differentiating its position with respect to time:

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2.$$

The velocity of the second mass depends on the motion of both rods. Differentiating its position yields

$$v_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2).$$

Substituting these expressions into the kinetic energy gives

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right).$$

The presence of the cross-term proportional to $\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$ reflects the coupling between the two degrees of freedom. This coupling is responsible for the transfer of energy between the pendulums and plays a central role in the emergence of chaotic motion.

c) *Potential energy, Lagrangian, and Equations of Motion*

The potential energy of the system arises from gravity. Choosing the zero of potential energy at the pivot point, the gravitational potential energy is

$$V = -m_1 g l_1 \cos\theta_1 - m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right).$$

The Lagrangian of the double pendulum is then constructed as

$$\mathcal{L} = T - V,$$

where T and V are the kinetic and potential energies derived above.

Applying the Euler–Lagrange equations to the generalised coordinates θ_1 and θ_2 yields two coupled, nonlinear second-order differential equations. These equations fully determine the dynamics of the system, but their complexity prevents the existence of a closed-form analytical solution. The nonlinearity arises from both the trigonometric dependence on the angles and the coupling between the angular velocities.

As a result, while the equations of motion are deterministic and derived exactly from first principles, practical prediction of the system's motion requires numerical integration. This transition from analytical formulation to numerical analysis highlights both the power and the limitations of classical mechanics when applied to chaotic systems such as the double pendulum.

VI. Prediction, Chaos, and Numerical Methods

a) Deterministic Chaos in the Double Pendulum

Although the equations of motion of the double pendulum are derived deterministically from the Lagrangian formulation, the system exhibits chaotic behaviour for a wide range of initial conditions. Chaos in this context refers to extreme sensitivity to initial conditions, whereby arbitrarily small differences in starting angles or angular velocities lead to exponentially diverging trajectories over time. This sensitivity does not arise from randomness or external noise, but from the nonlinear structure of the governing equations themselves.

In the double pendulum, chaos emerges due to the strong coupling between the two degrees of freedom and the presence of nonlinear trigonometric terms in the equations of motion. Energy is continuously exchanged between the two pendulums in an irregular manner, causing the system to transition unpredictably between rotational and oscillatory motion. As a result, even when initial conditions are specified with high precision, long-term behaviour becomes practically impossible to predict.

It is important to distinguish determinism from predictability. The motion of the double pendulum is fully deterministic in the sense that its future evolution is uniquely determined by its initial conditions and equations of motion. However, because any physical measurement of initial conditions is subject to finite precision, prediction beyond a limited time horizon becomes unreliable. This distinction is central to understanding the limits of classical mechanics when applied to chaotic systems.

b) Numerical Integration of the Equations of Motion

Because the equations of motion of the double pendulum do not admit closed-form solutions, numerical methods are required to compute the system's trajectory. These methods approximate the continuous evolution of the system by discretising time into small steps and iteratively updating the state variables. One commonly used technique is the fourth-order Runge–Kutta (RK4) method, which provides a balance between computational efficiency and numerical accuracy.

In numerical simulations, the state of the system at a given time is specified by the angles θ_1, θ_2 and their corresponding angular velocities. Given these initial conditions, the equations of motion derived from the Lagrangian are used to calculate the time derivatives required for numerical integration. The accuracy of the resulting trajectory depends critically on the chosen time step: excessively large steps introduce significant numerical error, while extremely small steps increase computational cost.

An important consideration in numerical simulations is the conservation of energy. Since the double pendulum is a conservative system, its total mechanical energy should remain constant over time. Deviations from energy conservation provide a useful diagnostic for assessing numerical accuracy. Even when energy is approximately conserved, however, chaotic divergence can still occur due to sensitivity to initial conditions rather than numerical instability.

c) Limits of Predictability

The combination of chaotic dynamics and numerical approximation imposes fundamental limits on the predictability of the double pendulum's motion. Small numerical errors, arising from discretisation and floating-point rounding, grow exponentially over time in a chaotic system. This means that two simulations starting from nearly identical initial conditions will eventually diverge, even when using accurate numerical methods.

These limitations are not failures of the Lagrangian formulation or of numerical integration techniques. Instead, they reflect intrinsic properties of chaotic systems. While short-term predictions can be made reliably, long-term prediction is fundamentally constrained by both physical measurement precision and the mathematical structure of the equations of motion.

The double pendulum, therefore, serves as a clear example of how exact classical laws can coexist with practical unpredictability. The Lagrangian formulation enables precise derivation of the governing equations and supports accurate short-term numerical prediction, while simultaneously revealing the inherent limits of predictability imposed by chaos.

VII. Hamiltonian vs Lagrangian

The Lagrangian and Hamiltonian formulations of classical mechanics provide mathematically equivalent descriptions of physical systems, yet they emphasise different aspects of dynamics. While the Lagrangian formulation expresses motion in terms of generalised coordinates and velocities, $(q, \dot{q})$, the Hamiltonian formulation reformulates the same dynamics using generalised coordinates and conjugate momenta (q, p). This change of variables leads to important conceptual and practical differences, particularly in the analysis of complex and chaotic systems.

The Hamiltonian is obtained from the Lagrangian through a Legendre transformation,

$$H = \sum_i p_i \dot{q}_i - \mathcal{L},$$

where the conjugate momenta are defined as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

For conservative mechanical systems in which the Lagrangian does not depend explicitly on time, the Hamiltonian corresponds to the total mechanical energy of the system, $H = T + V$. It is important to note that this equality is a consequence of the structure of such systems rather than a general definition of the Hamiltonian.

A key distinction between the two formulations lies in their treatment of phase space. The Hamiltonian framework describes motion as trajectories in phase space, where both positions and momenta evolve according to Hamilton's equations. This perspective is particularly useful for analysing conservation laws, stability, and the geometric structure of dynamical systems. In the context of the double pendulum, the Hamiltonian formulation provides insight into energy conservation and the complex phase-space structure associated with chaotic motion.

Despite these advantages, the Lagrangian formulation remains more practical for deriving equations of motion for constrained systems. Generalised coordinates allow constraints to be incorporated directly, simplifying the derivation process. For this reason, the Lagrangian approach is employed in this paper to obtain the equations of motion of the double pendulum, while the Hamiltonian formulation is introduced primarily to provide conceptual context.

Together, the Lagrangian and Hamiltonian formulations illustrate how the same physical laws can be expressed through different mathematical structures, each offering distinct insights. The equivalence of these formulations highlights the depth of classical mechanics and underscores the relevance of analytical mechanics to modern physics, including the foundations of quantum mechanics and chaos theory.

VIII. Euler–Lagrange vs "regular" Lagrangian

In discussions of analytical mechanics, the Euler–Lagrange equations are sometimes mistakenly treated as a distinct form of the Lagrangian itself. In reality, the Euler–Lagrange equations are not an alternative Lagrangian, but rather the equations of motion derived from a given Lagrangian through the application of the principle of stationary action.

Starting from the action $S = \int \mathcal{L}\, dt$, the requirement that the action be stationary with respect to small variations in the generalised coordinates leads directly to the Euler–Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

These equations provide a general prescription for obtaining the equations of motion once the Lagrangian of a system has been specified.

The distinction is conceptually important. The Lagrangian encapsulates the physical properties of the system through its energy structure, while the Euler–Lagrange equations represent the mathematical consequences of requiring the action to be stationary. Different systems may share similar Euler–Lagrange equations in form, yet correspond to entirely different physical Lagrangians.

Clarifying this relationship helps prevent misunderstandings and reinforces the logical structure of Lagrangian mechanics. In the context of this paper, the Euler–Lagrange equations serve as the formal link between the Lagrangian formulation introduced earlier and the explicit equations of motion used for numerical prediction of the double pendulum's dynamics.

IX. Application in aerospace engineering

The Lagrangian formulation of mechanics plays a central role in many areas of aerospace engineering, particularly in the modelling and analysis of systems involving multiple interconnected components and constraints. Aerospace systems often consist of rigid bodies linked by joints, subject to gravitational and inertial forces, making energy-based formulations especially useful.

One important application is spacecraft attitude dynamics. Satellites and space vehicles rotate freely in three dimensions and may include articulated components such as solar panels, antennas, or deployable booms. These systems possess multiple degrees of freedom and are subject to geometric constraints similar to those found in the double pendulum. The Lagrangian formulation allows the equations of motion for such systems to be derived systematically using generalised coordinates, without the need to explicitly calculate internal constraint forces.

Lagrangian mechanics is also widely used in the analysis of multi-body flight systems, including launch vehicles and space stations. During ascent or manoeuvring, components of these systems interact dynamically, exchanging energy in complex ways. Numerical integration of equations derived from Lagrangian models is therefore essential for simulating system behaviour and ensuring stability and control.

The connection between Lagrangian mechanics and numerical simulation is particularly relevant in aerospace engineering, where prediction over limited time scales is often sufficient for guidance, navigation, and control. As demonstrated by the double pendulum, even deterministic systems can exhibit sensitivity to initial conditions, reinforcing the importance of robust numerical methods and control strategies.

By providing a structured approach to modelling constrained, multi-degree-of-freedom systems, the Lagrangian formulation underpins much of modern aerospace dynamics. The study of chaotic systems, such as the double pendulum, therefore offers valuable insight into both the power and the limitations of classical mechanics in applied engineering contexts.

X. Conclusion

This paper has examined how the Lagrangian formulation of classical mechanics can be used to derive and numerically predict the motion of a double pendulum, a system that is both deterministic and chaotic. By reformulating the problem in terms of energy and generalised coordinates, the Lagrangian approach provides a systematic and efficient method for obtaining the equations of motion of a constrained multi-body system, avoiding the complications inherent in force-based Newtonian analyses.

Through the derivation of the coupled, nonlinear equations governing the double pendulum, it becomes clear that exact analytical solutions are not possible. Numerical integration, therefore, plays a central role in practical prediction. However, the chaotic nature of the system imposes fundamental limits on predictability: small uncertainties in initial conditions or numerical approximation grow rapidly over time, restricting reliable prediction to short time scales. This highlights the important distinction between deterministic laws and predictable outcomes.

The comparison between Lagrangian and Hamiltonian formulations further illustrates how different mathematical frameworks can offer complementary insights into the same physical system, particularly in understanding energy conservation and phase-space structure. Beyond its theoretical significance, the Lagrangian formulation has direct relevance to applied fields such as aerospace engineering, where constrained, multi-degree-of-freedom systems must be modelled and simulated accurately.

Ultimately, the double pendulum serves as a clear demonstration of both the power and the limitations of classical mechanics. The Lagrangian formulation enables precise modelling and short-term prediction while simultaneously revealing the intrinsic unpredictability of chaotic systems, underscoring the necessity of both analytical rigour and numerical methods in modern physics.